

LAGRANGIAN FLUID MECHANICS

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Abstract

The method of Lagrangians with covariant derivative (MLCD) is applied to a special type of Lagrangian density depending on scalar and vector fields as well as on their first covariant derivatives. The corresponding Euler-Lagrange's equations and energy-momentum tensors are found on the basis of the covariant Noether's identities.

1 Introduction

In the recent years it has been shown that every classical field theory [1], [2] could be considered as a theory of a continuous media with its kinematic characteristics. On the other side, "perfect fluids are equivalent to elastic media when the later are homogeneous and isotropic" [3]. The theory of fluids is usually based on a state equation and on a variational principle [4], [5], [6] with a given Lagrangian density depending on the variables constructing the equation of state (rest mass density, entropy, enthalpy, temperature, pressure etc.) and the 4-velocity of the points in the fluid. There are two canonical types of description of a fluid: the Eulerian picture and the Lagrangian picture. In the Eulerian picture, a point of the fluid and its velocity are identified with the state and the velocity of an observer moving with it. This means that every point in the fluid has a velocity which coincides with the velocity of an observer moving with it. In the Lagrangian picture, the observer has different velocity from the velocity of a point of the fluid observed by him. He observes the projections of the velocities of the points of the fluids from its own frame of reference. This means that an observer does not move with the points of the fluid but measures their velocities projecting them on its own velocity and standpoint. In other

words, the Eulerian picture is related to the description of a fluid by an observer moving locally with its points, and the Lagrangian picture is related to the description of a fluid by an observer not moving locally with its points. In the Eulerian picture the velocity of the points (the velocity of an observer) in a fluid appears as a field variable in the Lagrangian density describing the fluid. In the Lagrangian picture the velocity of an observer could be considered as an element of the projection formalism [called $(n+1) - 1$ - formalism, $\dim M = n$, where M is a differential manifold with dimension n , considered as a model of the space ($n = 3$) or the space-time ($n = 4$)]. The Lagrangian invariant L by the use of which a Lagrangian density \mathbf{L} is defined as $\mathbf{L} = \sqrt{-d_g} \cdot L$, where $d_g = \det(g_{ij})$. The tensor $g = g_{ij} \cdot dx^i \cdot dx^j = \frac{1}{2} \cdot g_{ij} \cdot (dx^i \otimes dx^j + dx^j \otimes dx^i)$ is the metric tensor field in M determining the possibility for introducing the square of the line element $ds^2 = g(d, d) = g_{ij} \cdot dx^i \cdot dx^j$ with $d = dx^i \cdot \partial_i \in T(M) := \cup_x T_x(M)$ [$T_x(M)$ is the tangent space to the point $x \in M$, $:=$ means by definition, $i, j, k, \dots = 1, 2, \dots, n$], interpreted as the infinitesimal distance between two points: point x with co-ordinates x^i and point \bar{x} with co-ordinates $x^i + dx^i$. The differentiable manifold M is considered here as a space with contravariant and covariant affine connections (whose components differ not only by sign) and metrics, i. e. $M \equiv (\bar{L}_n, g)$ -space [7], [8]. In (\bar{L}_n, g) -spaces with $n = 4$ the Lagrangian invariant L could be interpreted as the pressure p in a material system. This interpretation is based on the fact that for $n = 4$, the invariant L has the same dimension as the pressure $p : [p] = \frac{[\text{force}]}{[2\text{-surface}]} = [L]$ of a physical system, described by its Lagrangian density \mathbf{L} , i. e. $\mathbf{L} = \sqrt{-d_g} \cdot p$. It allows us to consider L as the pressure of a fluid. On this basis, we could formulate the following hypothesis:

Hypothesis. The pressure p of a dynamical system could be identified with the Lagrangian invariant L of this system, i. e.

$$p := L(g_{ij}, g_{ij;k}, g_{ij;k;l}, V^A{}_B, V^A{}_{B;i}, V^A{}_{B;i;j}) , \quad (1)$$

where g_{ij} are the components of the metric tensor field of the space (or space-time), where the system exists and $V^A{}_B$ are components of tensor fields describing the state of this system. $g_{ij;k}, g_{ij;k;l}$ are the first and second covariant derivative of the metric tensor g with respect to a covariant affine connection P and $V^A{}_{B;i}, V^A{}_{B;i;j}$ are the first and second covariant derivatives of the tensor fields $V \in \otimes^k_l(M)$ with respect to a contravariant affine connection Γ and a covariant affine connection P .

The hypothesis could be considered from two different points of view:

1. If the pressure in a dynamical system is given, i. e. if a state equation of the type

$$p := p(g_{ij}, g_{ij;k}, g_{ij;k;l}, V^A{}_B, V^A{}_{B;i}, V^A{}_{B;i;j})$$

is given, then p could be identified with the Lagrangian invariant L of the system. On this basis a Lagrangian theory of a system with a given pressure p could be worked out.

2. If a Lagrangian invariant L of a Lagrangian density $\mathbf{L} = \sqrt{-d_g} \cdot L$ is given, i.e. if

$$L := L(g_{ij}, g_{ij;k}, g_{ij;k;l}, V^A{}_B, V^A{}_{B;i}, V^A{}_{B;i;j})$$

is given, then L could be identified with the pressure p of the dynamical system. On these grounds a Lagrangian theory of fluids could be worked out with a given pressure p . In this case $L := p$.

Therefore, we can distinguish two cases:

Case 1. $L := p$ with given $p \in C^r(M)$.

Case 2. $p := L$ with given $L \in C^r(M)$.

In the present paper we consider a Lagrangian fluid mechanics by the use of the method of Lagrangians with covariant derivatives of the type (1). The Lagrangian invariant $L := p$ (considered as the pressure p) could depend on the velocity vector field $u \in T(M)$ of the points in the fluid, on other vector fields $\xi \in T(M)$ orthogonal (or not orthogonal) to u [$g(u, \xi) = 0$ or $g(u, \xi) \neq 0$], on scalar fields $f_N = f_N(x^k)$ ($N = 1, 2, \dots, m \in \mathbf{R}$) describing the state of the fluid as well as on the (first) covariant derivatives of the vector fields u and ξ . For a given in its explicit form Lagrangian density \mathbf{L} and its Lagrangian invariant L respectively, we find the covariant Euler-Lagrange's equations and the corresponding energy-momentum tensors.

2 Lagrangian density and Lagrangian invariant

Let us consider a Lagrangian density of the type $\mathbf{L} = \sqrt{-d_g} \cdot L$ with Lagrangian invariant L in the form:

$$\begin{aligned} L : &= p := p_0 + a_0 \cdot \rho \cdot e + b_0 \cdot g(\nabla_u(\rho \cdot u), u) + c_0 \cdot g(\nabla_u(\rho \cdot \xi), \xi) \quad (2) \\ &+ f_0 \cdot g(\nabla_u(\rho \cdot \xi), u) + h_0 \cdot g(\nabla_u(\rho \cdot u), \xi) \\ &+ \kappa_0 \cdot \rho \cdot \frac{m_0}{\sqrt{g(\xi, \xi)}} + k_0 \cdot \frac{M_0 \cdot \rho}{[g(u, u)]^k \cdot [g(\xi, \xi)]^m} \\ &+ k_1 \cdot \rho^r + p_1(f_N, f_{N,i}, u^i) \quad . \end{aligned}$$

The quantities $p_0, a_0, b_0, c_0, f_0, h_0, \kappa_0, k_0, m_0, M_0$, and k_1 are constants, k, m, r are real numbers, $f_N := f_N(x^k) \in C^r(M)$ are real functions identified as thermodynamical and kinematical variables, $N \in \mathbf{N}$. The function $\rho = \rho(x^k)$ is an invariant function with respect to the co-ordinates in M . The vector field $u = u^i \cdot \partial_i \in T(M)$ is a contravariant non-isotropic (non-null) vector field with $g(u, u) := e \neq 0$. The vector field $\xi \in T(M)$ is a contravariant vector field with $g(\xi, \xi) \neq 0$ in the cases when $\kappa_0 \neq 0, m_0 \neq 0, k_0 \neq 0, M_0 \neq 0, \rho(x^k) \neq 0$ and $g(\xi, \xi) = 0$ or $g(u, \xi) = 0$ if $\kappa_0 = m_0 = k_0 = M_0 = 0$. The constants $a_0, b_0, c_0, f_0, h_0, \kappa_0, k_0$, and k_1 could also be considered as Lagrangian multipliers to the corresponding constraints of 1. kind:

$$a_0 : \rho \cdot e = 0, \quad (3)$$

$$\begin{aligned}
b_0 & : g(\nabla_u(\rho \cdot u), u) = 0 , \\
c_0 & : g(\nabla_u(\rho \cdot \xi), \xi) = 0 , \\
f_0 & : g(\nabla_u(\rho \cdot \xi), u) = 0 , \\
h_0 & : g(\nabla_u(\rho \cdot u), \xi) = 0 , \\
\kappa_0 & : \rho \cdot \frac{m_0}{\sqrt{g(\xi, \xi)}} = 0 , \\
k_0 & : \frac{M_0 \cdot \rho}{[g(u, u)]^k \cdot [g(\xi, \xi)]^m} = 0 , \\
k_1 & : \rho^r = 0 \rightarrow \rho = 0 .
\end{aligned}$$

Depending on the considered case the corresponding constants could be chosen to be or not to be equal to zero.

2.1 Representation of the Lagrangian invariant L in a useful for variations form

For finding out the Euler-Lagrange's equations one needs to represent the Lagrangian invariant L in a form, suitable for the application of the method of Lagrangians with covariant derivatives [9]. For this reason the pressure $p = L$ could be written in the form

$$p = p_0 + p_1(f_N, u f_N) + k_1 \cdot \rho^r + \rho \cdot f + (u\rho) \cdot b , \quad (4)$$

where

$$\begin{aligned}
f & : = a_0 \cdot e + b_0 \cdot g(a, u) + c_0 \cdot g(\nabla_u \xi, \xi) + f_0 \cdot g(\nabla_u \xi, u) + h_0 \cdot g(a, \xi) \\
& \quad + \kappa_0 \cdot \frac{m_0}{\sqrt{g(\xi, \xi)}} + k_0 \cdot \frac{M_0}{[g(u, u)]^k \cdot [g(\xi, \xi)]^m} , \\
b & : = b_0 \cdot e + c_0 \cdot g(\xi, \xi) + (f_0 + h_0) \cdot l , \\
a & : = \nabla_u u = u^i{}_{;j} \cdot u^j = a^i \cdot \partial_i , \quad l := g(u, \xi) .
\end{aligned} \quad (5)$$

In a co-ordinate basis f , b , a , and l have the form:

$$\begin{aligned}
f & = g_{\overline{kl}} \cdot [a_0 \cdot u^k \cdot u^l + b_0 \cdot u^k{}_{;m} \cdot u^m \cdot u^l + c_0 \cdot \xi^k{}_{;m} \cdot u^m \cdot \xi^l \\
& \quad + f_0 \cdot \xi^k{}_{;m} \cdot u^m \cdot u^l + h_0 \cdot u^k{}_{;m} \cdot u^m \cdot \xi^l] \\
& \quad + \kappa_0 \cdot \frac{m_0}{\sqrt{g_{\overline{kl}} \cdot \xi^k \cdot \xi^l}} + k_0 \cdot \frac{M_0}{[g_{\overline{kl}} \cdot u^k \cdot u^l]^k \cdot [g_{mn} \cdot \xi^m \cdot \xi^n]^m} , \\
b & = g_{\overline{kl}} \cdot [b_0 \cdot u^k \cdot u^l + c_0 \cdot \xi^k \cdot \xi^l + (f_0 + h_0) \cdot u^k \cdot \xi^l] .
\end{aligned} \quad (6)$$

Therefore, we can consider p , f , and b as functions of the field variables f_N , ρ , u , ξ , and g as well as of their corresponding first covariant derivatives.

3 Euler-Lagrange's equations for the variables on which the pressure p depends

We can apply now the method of Lagrangians with covariant derivatives to the explicit form of the pressure p and find the Euler-Lagrange's equations for the variables f_N , ρ , u , ξ , and g . After long (but not so complicated computations) the Euler-Lagrange's equations follow in the form:

1. Euler-Lagrange's equations for the thermodynamical functions f_N :

$$\frac{\partial p_1}{\partial f_N} - \left(\frac{\partial p_1}{\partial f_{N,i}} \right)_{;i} + q_i \cdot \frac{\partial p_1}{\partial f_{N,i}} = 0 , \quad (7)$$

where

$$\begin{aligned} q_i &= T_{ki}{}^k - \frac{1}{2} \cdot g^{\overline{kl}} \cdot g_{kl;i} + g_k^l \cdot g_{l;i}^k , \\ T_{ki}{}^k &= g_l^k \cdot T_{ki}{}^l , \quad T_{ki}{}^l = \Gamma_{ik}^l - \Gamma_{ki}^l , \quad g_{l;i}^k = \Gamma_{li}^k + P_{li}^k . \end{aligned} \quad (8)$$

2. Euler-Lagrange's equation for the function ρ :

$$ub = k_1 \cdot r \cdot \rho^{r-1} + f + (q - \delta u) \cdot b , \quad (9)$$

where

$$ub = u^k \cdot b_{,k} , \quad q = q_i \cdot u^i , \quad \delta u = u^i{}_{;i} = u^i{}_{;k} \cdot g_i^k . \quad (10)$$

3. Euler-Lagrange's equations for the contravariant vector field u :

$$\begin{aligned} (h_0 - f_0) \cdot \xi^i{}_{;k} \cdot u^k &= g^{ij} \cdot [b \cdot (\log \rho)_{,j} + \frac{1}{\rho} \cdot \frac{\partial p_1}{\partial u^j}] \\ &+ \{ 2 \cdot (a_0 - k_0 \cdot k \cdot \frac{M_0}{[g(u, u)]^{k+1} \cdot [g(\xi, \xi)]^m}) \\ &+ b_0 \cdot [q - \delta u + u(\log \rho)] \} \cdot u^i \\ &+ \{ h_0 \cdot (q - \delta u) + f_0 \cdot [u(\log \rho)] \} \cdot \xi^i \\ &+ g_{kl}^{\overline{}} \cdot [(b_0 \cdot u^l + h_0 \cdot \xi^l) \cdot (u^k{}_{;j} - g_{j;m}^k \cdot u^m) \cdot g^{ji} \\ &+ (c_0 \cdot \xi^l + f_0 \cdot u^l) \cdot \xi^k{}_{;j} \cdot g^{ji}] \\ &- g^{ij} \cdot g_{j\overline{k};m} \cdot u^m \cdot (b_0 \cdot u^k + h_0 \cdot \xi^k) , \end{aligned} \quad (11)$$

where

$$g_{j\overline{k};m}^{\overline{}} := f^n{}_{;j} \cdot f^l{}_{;k} \cdot g_{nl;m} , \quad u(\log \rho) = u^i \cdot (\log \rho)_{,i} . \quad (12)$$

4. Euler-Lagrange's equations for the contravariant vector field ξ :

$$\begin{aligned} (f_0 - h_0) \cdot a^i &= [f_0 \cdot (q - \delta u) + h_0 \cdot u(\log \rho)] \cdot u^i \\ &+ \{ c_0 \cdot [q - \delta u + u(\log \rho)] \\ &- (\kappa_0 \cdot \frac{m_0}{[g(\xi, \xi)]^{3/2}} + 2 \cdot k_0 \cdot m \cdot \frac{M_0}{[g(u, u)]^k \cdot [g(\xi, \xi)]^{m+1}} \} \cdot \xi^i \\ &- g^{ij} \cdot g_{j\overline{k};m} \cdot u^m \cdot (c_0 \cdot \xi^k + f_0 \cdot u^k) \\ &- g^{ij} \cdot g_{j;m}^k \cdot u^m \cdot g_{kl}^{\overline{}} \cdot (c_0 \cdot \xi^l + f_0 \cdot u^l) . \end{aligned} \quad (13)$$

5. Euler-Lagrange's equations for the covariant metric tensor g :

$$\begin{aligned}
g^{ij} &= -\frac{2}{p} \cdot \{ [b_0 \cdot (u\rho) + \rho \cdot (a_0 - k_0 \cdot k \cdot \frac{M_0}{[g(u, u)]^{k+1} \cdot [g(\xi, \xi)]^m})] \cdot u^i \cdot u^j \\
&\quad + [c_0 \cdot (u\rho) - \rho \cdot (\frac{\kappa_0}{2} \cdot \frac{m_0}{[g(\xi, \xi)]^{3/2}} + k_0 \cdot m \cdot \frac{M_0}{[g(u, u)]^k \cdot [g(\xi, \xi)]^{m+1}})] \xi^i \cdot \xi^j \\
&\quad + \frac{1}{2} \cdot \rho \cdot [b_0 \cdot (a^i \cdot u^j + a^j \cdot b^i) + h_0 \cdot (a^i \cdot \xi^j + a^j \cdot \xi^i) \\
&\quad + c_0 \cdot (\xi^i{}_{;m} \cdot \xi^j + \xi^j{}_{;m} \cdot \xi^i) \cdot u^m + f_0 \cdot (\xi^i{}_{;m} \cdot u^j + \xi^j{}_{;m} \cdot u^i) \cdot u^m] \\
&\quad + \frac{1}{2} \cdot (u\rho) \cdot (f_0 + h_0) \cdot (u^i \cdot \xi^j + u^j \cdot \xi^i) \} \\
&= -\frac{2}{p} \cdot [A \cdot u^i \cdot u^j + B \cdot \xi^i \cdot \xi^j + \frac{1}{2} \cdot \rho \cdot C^{ij} + \frac{1}{2} \cdot (u\rho) \cdot D^{ij}] ,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
A &= b_0 \cdot (u\rho) + \rho \cdot (a_0 - k_0 \cdot k \cdot \frac{M_0}{[g(u, u)]^{k+1} \cdot [g(\xi, \xi)]^m}) , \\
B &= c_0 \cdot (u\rho) - \rho \cdot (\frac{\kappa_0}{2} \cdot \frac{m_0}{[g(\xi, \xi)]^{3/2}} + k_0 \cdot m \cdot \frac{M_0}{[g(u, u)]^k \cdot [g(\xi, \xi)]^{m+1}}) , \\
C^{ij} &= b_0 \cdot (a^i \cdot u^j + a^j \cdot b^i) + h_0 \cdot (a^i \cdot \xi^j + a^j \cdot \xi^i) \\
&\quad + c_0 \cdot (\xi^i{}_{;m} \cdot \xi^j + \xi^j{}_{;m} \cdot \xi^i) \cdot u^m + f_0 \cdot (\xi^i{}_{;m} \cdot u^j + \xi^j{}_{;m} \cdot u^i) \cdot u^m , \\
D^{ij} &= (f_0 + h_0) \cdot (u^i \cdot \xi^j + u^j \cdot \xi^i) .
\end{aligned} \tag{15}$$

The Euler-Lagrange's equations for the different variables are worth to be investigated in details in general as well as for every special case with a subset of constant different from zero.

3.1 Conditions for the pressure p which follow from the Euler-Lagrange's equations for the covariant metric field g

The Euler-Lagrange's equations (ELEs) for the metric field g lay down conditions to the form of the pressure p and its dependence on the other variables. The Euler-Lagrange's equations for g could be written in the general form as

$$\frac{\partial p}{\partial g_{ij}} + \frac{1}{2} \cdot g^{ij} = 0 . \tag{16}$$

After contracting with g_{ij} and summarizing over i and j we obtain the condition

$$\frac{\partial p}{\partial g_{ij}} \cdot g_{ij} + \frac{n}{2} \cdot p = 0 \quad p = -\frac{2}{n} \cdot \frac{\partial p}{\partial g_{ij}} \cdot g_{ij} . \tag{17}$$

On the other side, if we contract the ELEs for p with $\xi_j = g_{j\overline{m}} \cdot \xi^{\overline{m}}$ or with $u_j = g_{j\overline{n}} \cdot u^{\overline{n}}$ we obtain the following relations respectively:

$$(A^i{}_k - g^i_k) \cdot \xi^k = 0 , \quad (A^i_k - g^i_k) \cdot u^k = 0 , \tag{18}$$

where

$$A^i{}_k = -\frac{2}{p} \cdot \frac{\partial p}{\partial g_{lj}} \cdot g_{jm} \cdot f^m{}_k \cdot f_l{}^i \quad . \quad (19)$$

$f^m{}_k$ are components of the contraction tensor Sn [1] and $f^m{}_k \cdot f_m{}^i = g_k^i$. Since

$$-\frac{2}{p} \cdot \frac{\partial p}{\partial g_{ij}} \cdot g_{jk} = g_k^i \quad , \quad (20)$$

it follows that $A^i{}_k = g_k^i$ and therefore, the relations for ξ^i and u^i are identically fulfilled. The only condition remaining for p follows in the form

$$\begin{aligned} p = & \frac{2}{n+2} \cdot \{p_0 + p_1 + k_1 \cdot \rho^r \\ & + \rho \cdot [\frac{3}{2} \cdot \kappa_0 \cdot \frac{m_0}{\sqrt{g(\xi, \xi)}} + (k+m+1) \cdot k_0 \cdot \frac{M_0}{[g(u, u)]^k \cdot [g(\xi, \xi)]^m}]\} \quad . \end{aligned} \quad (21)$$

In the special case, when $n = 4$, we have

$$\begin{aligned} p = & \frac{1}{3} \cdot (p_0 + p_1 + k_1 \cdot \rho^r) \\ & + \rho \cdot [\frac{1}{2} \cdot \kappa_0 \cdot \frac{m_0}{\sqrt{g(\xi, \xi)}} + \frac{1}{3} \cdot (k+m+1) \cdot k_0 \cdot \frac{M_0}{[g(u, u)]^k \cdot [g(\xi, \xi)]^m}] \quad (22) \end{aligned}$$

By the use of the method of Lagrangians with covariant derivatives we can also find the corresponding energy-momentum tensors.

4 Energy-momentum tensors for a fluid with pressure p

The energy-momentum tensors for the given Lagrangian density \mathbf{L} could be found by the use of the method of Lagrangians with covariant derivatives on the basis of the covariant Noether's identities [9]

$$\begin{aligned} \overline{F}_i + \overline{\theta}_i{}^j{}_{;j} & \equiv 0 \quad \text{first Noether's identity,} \\ \overline{\theta}_i{}^j - {}_sT_i{}^j & \equiv \overline{Q}_i{}^j \quad \text{second Noether's identity.} \end{aligned} \quad (23)$$

One has to distinguish three types of energy-momentum tensors: (a) generalized canonical energy-momentum tensor $\overline{\theta}_i{}^j$; (b) symmetric energy-momentum tensor of Belinfante ${}_sT_i{}^j$, and (c) variational energy-momentum tensor of Euler-Lagrange $\overline{Q}_i{}^j$. All three energy-momentum tensors obey the second Noether's identity. After long computations we can find the energy-momentum tensors.

4.1 Generalized canonical energy-momentum tensor

The generalized energy-momentum tensor $\overline{\theta}_i{}^j$ could be obtained in the form

$$\overline{\theta}_i{}^j = \frac{\partial p_1}{\partial f_{N,j}} \cdot f_{N,i} + b \cdot \rho_{,i} \cdot u^j$$

$$\begin{aligned}
& +\rho \cdot g_{\overline{lm}} \cdot [(b_0 \cdot u^n + h_0 \cdot \xi^n) \cdot u^l_{;i} \cdot u^j + (c_0 \cdot \xi^n + f_0 \cdot u^n) \cdot \xi^l_{;i} \cdot u^j] \\
& +g_{\overline{ik}} \cdot \{(f_0 - h_0) \cdot (\xi\rho + \rho \cdot \delta\xi) \cdot u^j \cdot u^k - (f_0 - h_0) \cdot (u\rho + \rho \cdot \delta u) \cdot \xi^j \cdot u^k \\
& +\rho \cdot (h_0 - f_0) \cdot [(\xi^j \cdot a^k + b^j \cdot u^k) - (u^j \cdot d^k + u^k \cdot d^j)] \\
& +\rho \cdot (g_{\overline{ln};m} + g_{\overline{ln}} \cdot g_{l;m}^r) \cdot [(b_0 \cdot u^n + h_0 \cdot \xi^n) \cdot (g^{jl} \cdot u^k \cdot u^m - g^{lm} \cdot u^j \cdot u^k) \\
& +\frac{1}{2} \cdot (c_0 \cdot \xi^n + f_0 \cdot u^n) \cdot [g^{kl} \cdot (u^j \cdot \xi^m - u^m \cdot \xi^j) \\
& +g^{jl} \cdot (u^k \cdot \xi^m + u^m \cdot \xi^k) - g^{lm} \cdot (u^j \cdot \xi^k + u^k \cdot \xi^j)]]\} - p \cdot g_i^j,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
a^k &= u^k_{;l} \cdot u^l, & \delta u &= u^m_{;m}, & \delta\xi &= \xi^l_{;l}, & \xi\rho &= \rho_{;j} \cdot \xi^j, \\
b^j &= \xi^j_{;m} \cdot u^m, & d^k &= u^k_{;l} \cdot \xi^l, & f^k &= \xi^k_{;m} \cdot \xi^m.
\end{aligned} \tag{25}$$

4.2 Symmetric energy-momentum tensor of Belinfante

The symmetric energy-momentum tensor of Belinfante ${}_sT_i{}^j$ could be obtained in the form

$$\begin{aligned}
{}_sT_i{}^j &= g_{\overline{ik}} \cdot \{[b_0 \cdot (u\rho + \rho \cdot \delta u) + (f_0 - h_0) \cdot (\xi\rho + \rho \cdot \delta\xi)] \cdot u^j \cdot u^k \\
& +c_0 \cdot (u\rho + \rho \cdot \delta u) \cdot \xi^j \cdot \xi^k + h_0 \cdot (u\rho + \rho \cdot \delta u) \cdot (u^j \cdot \xi^k + u^k \cdot \xi^j) \\
& +\rho \cdot [b_0 \cdot (u^j \cdot a^k + u^k \cdot a^j) + c_0 \cdot (\xi^j \cdot b^k + \xi^k \cdot b^j) \\
& +f_0 \cdot (u^j \cdot d^k + u^k \cdot d^j) + h_0 \cdot (\xi^j \cdot a^k + \xi^k \cdot a^j) \\
& +h_0 \cdot (u^j \cdot b^k + u^k \cdot b^j) - h_0 \cdot (u^j \cdot d^k + u^k \cdot d^j)] \\
& +\rho \cdot (g_{\overline{ln};m} + g_{\overline{ln}} \cdot g_{l;m}^r) \cdot [(b_0 \cdot u^n + h_0 \cdot \xi^n) \cdot (g^{kl} \cdot u^j \cdot u^m \\
& +g^{jl} \cdot u^k \cdot u^m - g^{ml} \cdot u^j \cdot u^k) \\
& +\frac{1}{2} \cdot (c_0 \cdot \xi^n + f_0 \cdot u^n) \cdot [g^{kl} \cdot (u^j \cdot \xi^m + u^m \cdot \xi^j) \\
& +g^{jl} \cdot (u^k \cdot \xi^m + u^m \cdot \xi^k) - g^{lm} \cdot (u^j \cdot \xi^k + u^k \cdot \xi^j)]]\} - p \cdot g_i^j.
\end{aligned} \tag{26}$$

4.3 Variational energy-momentum tensor of Euler-Lagrange

The variational energy-momentum tensor of Euler-Lagrange $\overline{Q}_i{}^j$ could be obtained in the form

$$\begin{aligned}
\overline{Q}_i{}^j &= \frac{\partial p_1}{\partial u^i} \cdot u^j + b \cdot \rho_{;i} \cdot u^j \\
& +\rho \cdot g_{\overline{lm}} \cdot [(b_0 \cdot u^n + h_0 \cdot \xi^n) \cdot u^l_{;i} \cdot u^j + (c_0 \cdot \xi^n + f_0 \cdot u^n) \cdot \xi^l_{;i} \cdot u^j] \\
& -g_{\overline{ik}} \cdot \{b_0 \cdot (u\rho + \rho \cdot \delta u) \cdot u^j \cdot u^k + c_0 \cdot (u\rho + \rho \cdot \delta u) \cdot \xi^j \cdot \xi^k \\
& +f_0 \cdot (u\rho + \rho \cdot \delta u) \cdot \xi^j \cdot u^k + h_0 \cdot (u\rho + \rho \cdot \delta u) \cdot u^j \cdot \xi^k \\
& +\rho \cdot [b_0 \cdot (a^j \cdot u^k + a^k \cdot u^j) + c_0 \cdot (b^j \cdot \xi^k + b^k \cdot \xi^j) \\
& +f_0 \cdot (b^j \cdot u^k + a^k \cdot \xi^j) + h_0 \cdot (a^j \cdot \xi^k + b^k \cdot u^j)]
\end{aligned} \tag{27}$$

$$\begin{aligned}
& +\rho \cdot (g_{\overline{lm};m} + g_{\overline{rm}} \cdot g_{l;m}^r) \cdot [(b_0 \cdot u^n + h_0 \cdot \xi^n) \cdot g^{kl} \cdot u^j \cdot u^m \\
& + (c_0 \cdot \xi^n + f_0 \cdot u^n) \cdot g^{kl} \cdot u^m \cdot \xi^j] \} \quad .
\end{aligned}$$

From the explicit form of the energy-momentum tensors and the second Noether's identity the relation

$$\frac{\partial p_1}{\partial u^i} \cdot u^j = \frac{\partial p_1}{\partial f_{N,j}} \cdot f_{N,i} \quad . \quad (28)$$

follows.

5 Special cases

The general form of the Lagrangian density \mathbf{L} could be specialized for different from zero constants p_0 , a_0 , b_0 , c_0 , f_0 , h_0 , κ_0 , k_0 , m_0 , M_0 , and k_1 . If only p_0 , a_0 and h_0 are different from zero constants, then the corresponding Euler-Lagrange's equations and energy-momentum tensors describe a fluid with points moving on auto-parallel lines [10]. All more general cases are also worth to be investigated. This will be the task of another paper.

6 Conclusion

In the present paper Lagrangian theory for fluids over (\overline{L}_n, g) -spaces is worked out on the basis of the method of Lagrangians with covariant derivatives. A concrete Lagrangian density is proposed. The Euler-Lagrange's equations and the energy-momentum tensors are found. They could be used for considering the motion of fluids and their kinematic characteristics. It is shown that the description of fluids on the basis of the identification of their pressure with a Lagrangian invariant could simplify many problems in the fluids mechanics. On the other side, every classical field theory over spaces with affine connections and metrics could be considered as a concrete Lagrangian theory of a fluid with given pressure.

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